Local control of entanglement in a spin chain

Francesco Plastina[1] and Tony J. G. Apollaro[2]

[1] *Dip. Fisica, Università della Calabria, & INFN - Gruppo collegato di Cosenza, 87036 Arcavacata di Rende (CS) Italy*
[2] *Dip. Fisica, Università degli Studi di Firenze, Via G. Sansone 1, 50019 Sesto F.no (FI) Italy*
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In a ferromagnetic spin chain, the control of the local effective magnetic field allows to manipulate the static and dynamical properties of entanglement. In particular, the propagation of quantum correlations can be driven to a great extent so as to achieve an entanglement transfer on demand toward a selected site.

When the superposition principle is applied to an (at least) bi-partite system, highly non local and purely quantum correlations (entanglement) can appear among the parties. This constitutes a crucial resource for many applications in quantum communication[1]. Consequently, the problem of entanglement distribution has become of central interest: quantum correlations are generated by local interactions; therefore methods are required to transfer either the entangled particles or their state at a distance. It has been shown theoretically that spin chains are efficient quantum channels for short distance entanglement transfer [2]; thus the ability to manipulate the propagation of entanglement in a spin chain can be very important and it has already been shown that breaking the translational invariance of the chain can produce very interesting results in this respect [3, 4].

In this paper, we show that a control of both static and dynamical properties of entanglement can be achieved by acting locally to modify the level spacing of some qubits. In the magnetic language, we analyze a system subject to a spatially inhomogeneous magnetic field; that is, a system with 'diagonal defects' ('impurities'), [5].

In contrast to the usual case in which local actions cannot affect non-local physical quantities; here, due to the spin-spin interaction, the local control of the magnetic field modifies both the distribution of correlations in the ground state and the entanglement propagation along the chain. Indeed, spatial inhomogeneities of the external field lead to an Anderson-like localization of entanglement [6], and this can occur even for a single defect[7], giving rise to a mirror-like effect in the entanglement propagation [8]. Contrary to the usual description of localization phenomena, we do not conceive the impurity just as a bit of disorder in the system, but rather intend the modification of the local level spacing as a knob to *i)* control the content of "static" (ground state) entanglement, and *ii)* drive its propagation along the chain.

We consider a 1-D XX spin-$\frac{1}{2}$ closed chain, placed in an external magnetic field which is homogeneous everywhere but for two defect sites l_1 and l_2. This model is described by the Hamiltonian $H = H_0 + H_{def}$, with

$$H_0 = -\frac{\omega_0}{2} \sum_{i=-\frac{N}{2}}^{\frac{N}{2}} \sigma_z^i - J \sum_{i=-\frac{N}{2}}^{\frac{N}{2}} (\sigma_x^i \sigma_x^{i+1} + \sigma_y^i \sigma_y^{i+1}) \quad (1)$$

$$H_{def} = -\frac{1}{2} \sum_{j=1,2} \alpha_j \sigma_z^{l_j} \quad (2)$$

where ω_0 is the level spacing of each qubit, except for those residing at sites l_i, which have level separation $\omega_0 + \alpha_i$. Henceforth, we shall take $J = 1$ and use the ferromagnetic coupling constant as our energy unit (we have also set $\hbar = 1$). Furthermore, we set to zero the energy of the completely separable eigenstate $|0\rangle^{\otimes N}$, which is the unperturbed ground state for $\omega_0 > 1$, and which is still an energy eigenstate in the presence of defects.

This model can be solved exactly via the Jordan-Wigner (JW) transformation, which maps the spin chain into a spin-less fermion system [9]. However, we do not need the general solution here, since we will deal with states having at most one tilted spin (in the JW language, states lying in the single particle subspace), [10]. In the continuum limit $N \to \infty$, we solve the model (restricted to the single particle sector of its Hilbert space) using the Green operator technique [7].

To this end, we consider the Green operator G_0 describing the homogeneous chain. It is known [7, 8] that G_0 displays a branch cut on the real axis in the complex energy plane for energies $E \in [\omega_0 - 1, \omega_0 + 1]$. This cut signals a continuous energy band, which survives even in the presence of the defects.

Given G_0, the full (single particle) Green operator is

$$G = G_0 + G_0 T G_0, \quad (3)$$

where the T-matrix is given by

$$T = \frac{\sum_i |l_i\rangle t_i \langle l_i| + |l_2\rangle t_1 G_0(l_2, l_1) t_2 \langle l_1| + \text{h.c.}}{1 - t_1 t_2 G_0(l_1, l_2) G_0(l_2, l_1)}, \quad (4)$$

where $|n\rangle = |0\rangle^{\otimes N-1} \otimes |1\rangle_n$ is the state with one spin down (or, equivalently, one fermionic excitation) located at site n, and where the scattering coefficients are

$$t_i = -\frac{\alpha_i}{2 + \alpha_i G_0(l_i, l_i)}.$$

The T-matrix describes multiple scattering events of the single particle excitation at the two defects, and it is precisely the re-summation of the various scattering amplitudes which gives rise to the denominator in Eq. (4). The existence of zeros of this denominator is crucial since it implies that, besides the energy band discussed above, the model with defects displays some (at most two, in fact) discrete energy levels.

They lay above or below the energy band depending on the values of the defect fields, and their eigen-energies can be obtained analytically in some special cases. For example, for nearest neighbors defects, with distance $d = |l_2 - l_1| = 1$, by setting $x_{loc} = E_{loc} - \omega_0$ one gets

$$x_{loc} = \frac{\alpha_1\alpha_2\frac{\alpha_1+\alpha_2}{4} \pm \sqrt{\left[1 + \frac{(\alpha_1-\alpha_2)^2}{4}\right](1 - \frac{\alpha_1\alpha_2}{2})^2}}{1 - \alpha_1\alpha_2}$$

Only one of these two solutions (the one with the plus sign in front of the square root) exists if the defect strengths satisfy the relation $\alpha_1^{-1} + \alpha_2^{-1} \geq 1$. That is, the parameter space is divided in two regions, characterized by the number of discrete energy eigenvalues (which can be one or two).

If we restrict ourselves to the ordered phase of the unperturbed chain, $\omega_0 > 1$, and if we consider only $\alpha_i > 0$, the lowest (or the only existing) localized level becomes the fundamental state of the perturbed problem.

Many of these features obtained for nearest neighbors defects are generic and do not depend on their distance. From the analytic properties of the Green operator, [7], one can show that i) at most two localized states are present, whose position with respect to the energy band depends on the sign of the α's, ii) a region in the α_1-α_2 plane exists in which only one discrete state is found. However, this one-eigenstate region becomes thinner and thinner as d increases, iii) the localized states always have the structure of single excitation states of the form $|\psi_{loc}\rangle = \sum_n b_n |n\rangle$. The amplitudes b_n, obtained from the residues of G, can be expressed in terms of an inverse localization length $\xi = -\ln[-x_{loc} - \sqrt{x_{loc}^2 - 1}]$, as a function "bi-localized" around the two defects:

$$b_n = \text{const}\left(K_1 e^{-\xi|n-l_1|} + K_2 e^{-\xi|n-l_2|}\right). \quad (5)$$

The coefficients K_i are given by

$$K_i = \left(\frac{\alpha_i}{2\sqrt{x_{loc}^2 - 1} - \alpha_i}\right)^{\frac{1}{2}}.$$

Their ratio indicates the relative weights of the two localization region. For $\alpha_1 \gg \alpha_2$ one gets $|K_2| \ll |K_1|$ and the discrete levels are localized around l_1, while for $\alpha_1 \ll \alpha_2$ the localization center is l_2 since $|K_2| \gg |K_1|$. For equal defect strengths, $\alpha_1 = \alpha_2$, one finds $K_2 = \pm K_1$, where the upper (lower) sign refers to the lower (higher) of the
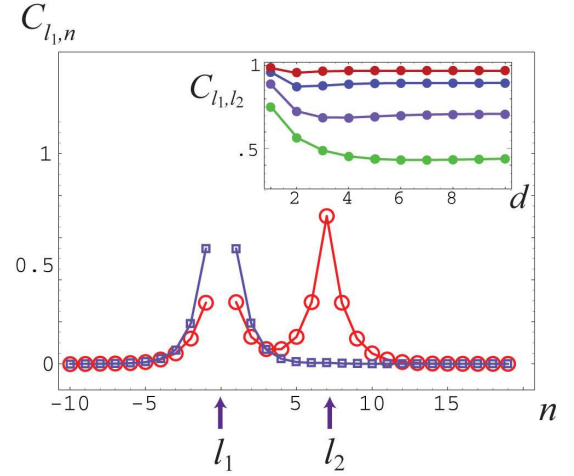

FIG. 1: (Color online) Ground state concurrence between the l_1-th and the n-th spin of the chain. The entanglement can be remotely controlled by changing the local field at the other defect. The line with open circles (boxes) corresponds to $\alpha_1 = \alpha_2 = 1.5$ ($\alpha_1 = 2, \alpha_2 = 1.5$). For equal defect strengths, spin l_1 is entangled both with its own neighborhood and with the other defect's one. In the asymmetric case, quantum correlations only survive within the localization region. The inset shows the concurrence between the two defects as a function of their distance for various values of defect fields. From below to above, the plots correspond to $\alpha_1 = \alpha_2 = 0.25; 0.5; 1; 2$.

two eigenstates. Thus, if $\alpha_1 = \alpha_2$, the two discrete states are given by equal-weight coherent superpositions of two localized parts centered on the two impurities. These states are highly entangled and display strong quantum correlations between the defects and their neighborhoods.

Even for quite small values of the defect fields, the localization length is smaller than the inter-site spacing (already $\alpha_1 = \alpha_2 = 1$ gives $\xi^{-1} < 1$, for any distance $d = |l_1 - l_2|$). This implies that the two discrete eigenstates are approximately given by the Bell combinations $(|01\rangle \pm |10\rangle)$, with the rest of the chain almost completely factorized in the state $|0\rangle$. It is noteworthy that this structure does not depend on the distance between the two defects, see the inset of Fig. 1, where the concurrence $C_{l_1 l_2} = 2|b_1 b_2|$ between the defects is shown as a function of $d = |l_1 - l_2|$. Thus, a long distance bi-partite entanglement can be obtained in the ground state of the chain. A similar behavior has been found in Ref. [11], with the difference that in our case this is a bulk property rather than a surface effect.

For generic values of the defect amplitudes, these states display entanglement between any pair of spins residing near each of the two defects, with the peculiarity that the pairwise entanglement for two spins around the same defect depends on the value of the local magnetic field at the other defect. This is illustrated in figure (1), where it is shown that the entanglement around l_1 is modified by changing the strength of local field at l_2, thus achieving
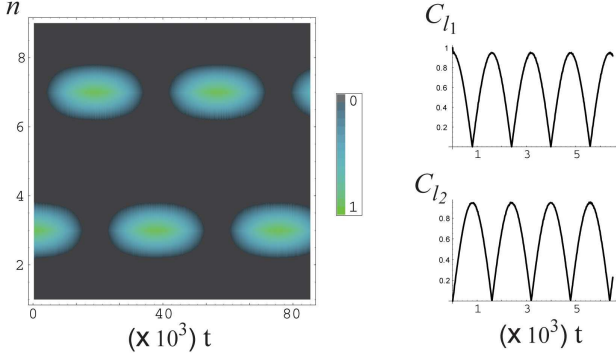
FIG. 2: (color online) Rabi oscillations of the entanglement between the two defects for the case $\alpha_1 = \alpha_2 = 1.5$.

a remote entanglement control.

The bi-local character of the discrete levels strongly affects the transport of entanglement along the chain. In particular, we consider the possibility of using the chain to send one partner of a maximally entangled pair. We assume that the spin at the sender site s is prepared in a singlet state with an external (un-coupled) qubit. The interaction between the spins causes a transfer of entanglement along the chain. Ideally, after a given transmission time t, one would like to get a singlet between the external qubit and the one residing at a receiving site r. To characterize the quality of the transmission, we evaluate the (final) concurrence between the external and the r-th qubits, denoted by $C_r(t)$.

This is given by $C_r(t) = |f_{s \rightarrow r}(t)|$, [2], where $f_{s \rightarrow r}(t)$ is the amplitude for the transfer of a fermionic excitation from site s to r. This is expressed in terms of the retarded Green operator as

$$f_{s \rightarrow r} = \sum_{E_{loc}} e^{-iE_{loc}t} b_r b_s^* + \int_{-\pi}^{\pi} d\theta e^{-iEt} g_r(E) g_s^*(E). \quad (6)$$

The first term describes transport mediated by localized states, while the second one gives a spin-wave mediated transfer, [12]. We analyze them separately.

The first contribution is effective only within a region of length ξ^{-1} around the two defect sites. This gives the noteworthy possibility of transmitting from one neighborhood to the other. This effects is illustrated in Fig. (2), where the entanglement is shown to jump from one defect to the other. Indeed, if the sender site coincides with one of the defects (say l_1), than the system evolves by performing almost perfect Rabi oscillations between the two discrete localized states, with a Rabi frequency given by $\omega_R = (E_{loc1} - E_{loc2})$. This can be understood by noticing that the initial singlet state between l_1 and the external qubit can be approximately written as

$$|\psi_{in}\rangle \simeq \frac{1}{\sqrt{2}} \Big[|0_{l_1}\rangle |1_{ext}\rangle - \frac{1}{\sqrt{2}} (|\psi_{loc1}\rangle + |\psi_{loc2}\rangle) |0_{ext}\rangle \Big],$$
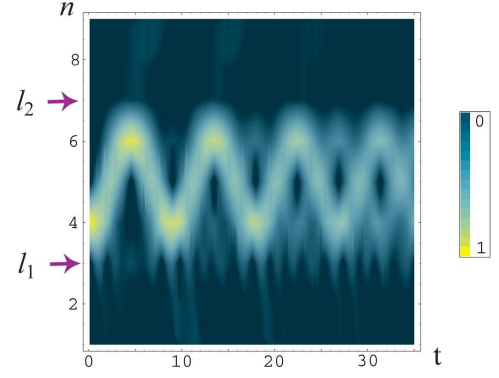

FIG. 3: (color online) Entanglement bouncing between the two defects which act like (non-perfect) mirrors. The location of the impurities is indicated by the two arrows and the defect-strengths are $\alpha_1 = \alpha_2 = 1.5$.

which implies that the concurrence between the l_i-th qubit and the external one is an harmonic function:

$$C_{l_1}(t) \simeq |\cos \omega_R t|, \qquad C_{l_2}(t) \simeq |\sin \omega_R t|. \quad (7)$$

The second term in Eq. (6) is an integral over the energy band, parameterized as $E = \omega_0 - \cos \theta$. It contains the state amplitudes of the continuous energy band, which can be written in terms of the retarded Green and T operators:

$$g_n(E) = \langle n| \Big[\mathbb{1} + G_0^+ T^+ \Big] |\psi_0(E)\rangle,$$

with the un-perturbed states such that $\langle n|\psi_0(E)\rangle = e^{in\theta}/\sqrt{2\pi}$. These states represent distorted spin waves of the system. They are the stationary scattering states of the single-particle Hamiltonian and can be constructed, starting from the usual magnon excitations, by including corrections due to multiple scattering at the defects. For moderate values of the α's, the distortion is not that big and the unperturbed plane wave nature can still be recognized. As a result, the energy eigenstates form (approximate) standing waves between the two defects. Their pattern is reflected in the entanglement propagation shown in Fig. (3) where the sender site is located between the two impurities which act as potential barriers for the spin waves, so that entanglement bounces back and forth between these two mirrors. The extreme situations is reached when the defects are next to nearest neighbors to each other, thus realizing an entanglement trap, see Fig. (4). Since the mirrors are not perfect, [8], the trapped entanglement decreases with time, the superimposed time oscillations being due to the $e^{-iE_{loc}t}$ factor of the symmetric discrete eigenstate (the only one that matters, in this case). Once these oscillations are subtracted, the short time behavior of the concurrence is found to be parabolic, with a convexity that decreases as the defect amplitudes are increased. A long-time (residual) trapped entanglement is also present, which is due
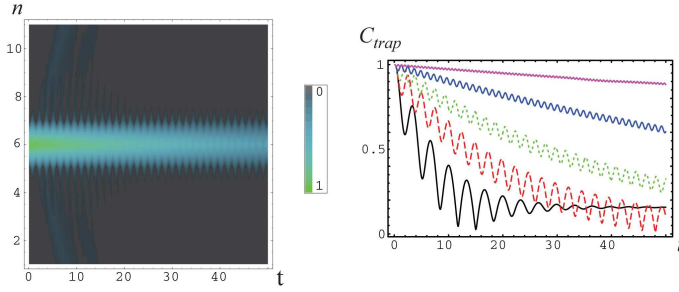
FIG. 4: (color online) Left: Entanglement trapping between the two defects in the case $\alpha_1 = \alpha_2 = 2$. Right: Time dependence of the trapped concurrence for $\alpha_1 = \alpha_2 = 0.5, 1, 1.5, 2.5, 5$ (from below to above).

to the tails of the localized state. It diminishes with increasing α's as the localization length does, and becomes negligible if ξ^{-1} is much smaller than the site spacing.

The Rabi oscillations illustrated above give a mean to transfer the entanglement reversibly between the defects. This method works independently of their distance, but has the drawback that the Rabi period increases with distance as $T \sim \alpha^d$ (this can be derived by perturbation theory for large α's, see Ref. [13]).

The form of the discrete states, however, suggests another, more effective method to achieve entanglement transfer on demand between the two defects; namely, the adiabatic passage (see [14] for a related proposal in which the coupling strength is varied instead of the local field).

The idea is to change the defect strengths adiabatically, so that the ground state of the system, having the general form given in Eq. (5), is adiabatically changed from the initial state $|\psi_{gs}(i)\rangle \simeq |l_1\rangle$ localized at l_1, to the final state $|\psi_{gs}(f)\rangle \simeq |l_2\rangle$, localized at l_2. This can be done by modifying the defect fields from the initial values

$$\alpha_1(i) \gg 1, \quad \alpha_2(i) \ll 1 \quad \Rightarrow \quad b_n(i) \simeq \delta_{n,l_1},$$

to the final (reversed) ones

$$\alpha_1(f) \ll 1, \quad \alpha_2(f) \gg 1 \quad \Rightarrow \quad b_n(f) \simeq \delta_{n,l_2}.$$

If this is done adiabatically, the system always remains in its instantaneous ground state, thus realizing the desired entanglement transfer provided the initial singlet state involves the external and the l_1-th qubit. To ensure the adiabaticity, the rates of change of the α's have to bo much smaller than the difference between the energies of the two discrete levels, $E_{21} = E_{loc2} - E_{loc1}$. The most dangerous point in this respect (i.e., the smallest E_{21}) occurs at the crossing, when $\alpha_1(t) = \alpha_2(t)$. But, since at this point $E_{21} \sim \alpha^d$, if the adiabatic procedure is designed such that the crossing occurs for a very small α, than the adiabaticity can be preserved even for transfer times much smaller than the Rabi period. This procedure has the additional advantage of effectively decoupling the

receiving site from the rest of the chain after the transfer has been performed due to its large final local field.

To summarize, we have discussed how to manipulate a spin chain with local control fields, showing that the static entanglement can be remotely controlled, and that entanglement propagation can be adjusted to a large extent in order to achieve transfer on demand. One possibility to implement this model in a realistic set up is to use the method proposed in Ref. [15] to "engineer" spin chains with atoms in an optical lattice. The addition of external static local electric or magnetic fields should enable the control of the qubit energy level spacing which is essential to test our proposal.

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